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NECESSARY AND SUFFICIENT CONDITIONS FOR QUASI-STRONG REGULARITY OF GRAPH PRODUCT

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Abstract. A k-regular graph ($k \geq 1$) with n vertices is called a quasi-strongly regular graph with parameter λ ($\lambda \in \mathbb{N}$) if any two adjacent vertices have exactly λ neighbors in common. A graph product is a binary operation on graphs. It is useful to describe graph as product of other primitive graphs. In this paper we present some necessary and sufficient conditions for Decartes product, Tensor product, Lexicographical product and Strong product to be quasi-strongly regular.

Keywords. Quasi-strongly regular graph, product graph.

1. INTRODUCTION

We consider in this paper only undirected and simple graphs. Let $G = (V, E)$ be a graph with the vertices set V and the edges set E. The neighborhood of a vertex $v \in V$, the set of adjacent vertices of v, is denoted by $N(v)$. If two vertices i and j are adjacent, then we write $i \sim j$. A quasi-strongly regular graph with parameters (n, k, λ) [8], denoted by qsrg(n, k, λ), is a k-regular graph on n vertices satisfying the condition: if $i \sim j$ then $\lambda = |N(i) \cap N(j)|$. The well-known Petersen graph (see Fig. 1) is a quasi-strongly regular graph qsrg$(10, 3, 0)$. The complete graph K_n is a quasi-strongly regular graph qsrg$(n, n-1, n-2)$. The disjoint union mK_a of m complete graphs K_a is a quasi-strongly regular graph qsrg$(ma, a-1, a-2)$, and its complement graph $\overline{mK_a}$ is a quasi-strongly regular graph with parameters $(ma, (m-1)a, (m-2)a)$.

Quasi-strongly regular graphs generalize a number of well-known classes, namely: strongly regular and distance regular graphs. A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that any two adjacent vertices have λ common neighbors and any two non-adjacent vertices have μ common neighbors. It was a very difficult problem to construct strongly regular graphs (see [1, 2, 3, 4, 5, 6]). In [9] we can find a list of strongly regular graphs on at most 64 vertices. The method to study strongly regular graphs are usually algebra combined with combinatorics [11].

The product of two graphs, namely the Cartesian product, has been studied first by Vizing [12]. This method is developed in [10] by Richard Hammack, Wilfried Imrich, Sandi Klavzar. All the products of two graphs with n_1 and n_2 vertices have exactly $n_1.n_2$ vertices. The set of edges is depends on the type of the graph product. Below we will present 4 types of graph products.

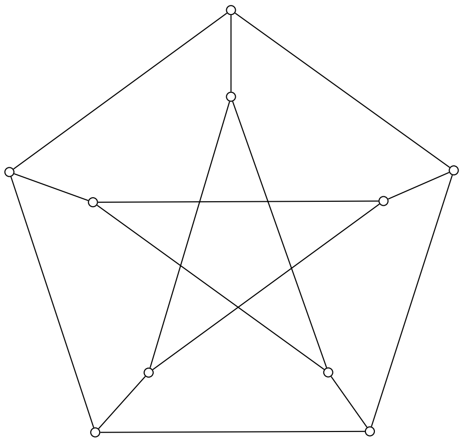

Figure 1. Petersen graph

Definition 1. (Decartes Product $G_1 \square G_2$) Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then the Decartes product graph $G = G_1 \square G_2$ is the graph with the vertex set

$$V_1 \times V_2 = \{(u_1, u_2) | u_1 \in V_1, u_2 \in V_2\}$$

and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are joined by an edge if $u_1 = v_1$ and $u_2 \sim v_2$ or if $u_1 \sim v_1$ and $u_2 = v_2$.

Let K_2 denote the complete graph with 2 vertices and C_4 denote the cycle of length 4, then Figure 2 shows the Decartes product graph $K_2 \square C_4$.

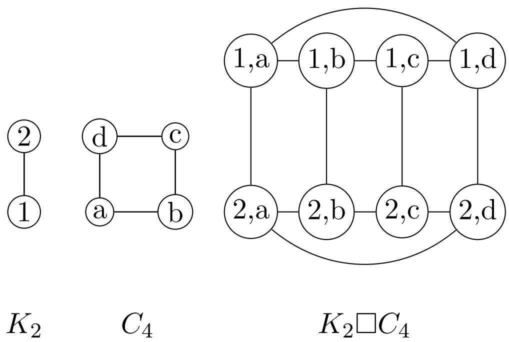

Figure 2. Decarstes product of K_2 and C_4

Definition 2. (Tensor product $G_1 \times G_2$) Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then the Tensor product graph $G = G_1 \times G_2$ is the graph with the vertex set

$$V_1 \times V_2 = \{(u_1, u_2) | u_1 \in V_1, u_2 \in V_2\}$$

and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are joined by an edge if $u_1 \sim v_1$ and $u_2 \sim v_2$.

Figure 3 shows the Tensor product graph $K_2 \times C_4$.

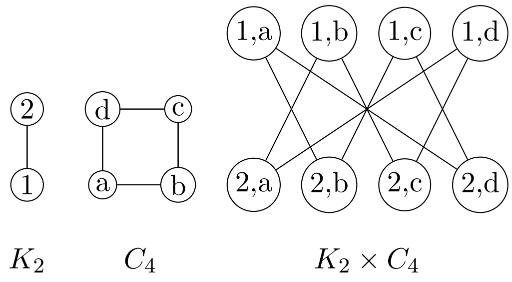

Figure 3. Tensor product of K_2 and C_4

Definition 3. (Lexicographical $G_1.G_2$) Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then the Lexicographical product graph $G = G_1.G_2$ is the graph with the vertex set

$$V_1 \times V_2 = \{(u_1, u_2) | u_1 \in V_1, u_2 \in V_2\}$$

and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are joined by an edge only if $(u_1 \sim v_1)$ or if $(u_1 = v_1) \wedge (u_2 \sim v_2)$.

Figure 4 shows the Lexicographical product graph $K_2.C_4$.

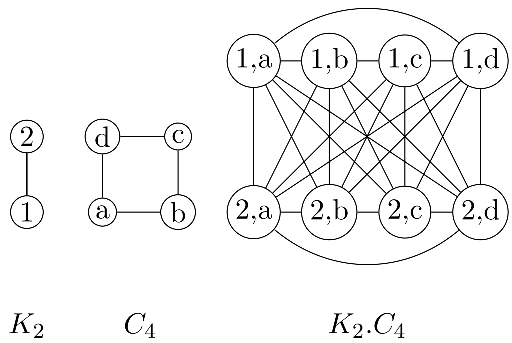

Figure 4. Lexicographical product of K_2 and C_4

Definition 4. (Strong product $G_1 \boxtimes G_2$) Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then the Strong product graph $G = G_1 \boxtimes G_2$ is the graph with the vertex set

$$V_1 \times V_2 = \{(u_1, u_2) | u_1 \in V_1, u_2 \in V_2\}$$

and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are joined by an edge only if $u_1 = v_1 \wedge u_2 \sim v_2$ or $u_1 \sim v_1 \wedge u_2 = v_2$ or $u_1 \sim v_1 \wedge u_2 \sim v_2$.

Figure 5 shows the Strong product graph $K_2 \boxtimes C_4$.

In what follows, we will prove some necessary and sufficient conditions for the quasi-strongly regularity of Decartes product, Tensor product, Lexicographical product and Strong product.

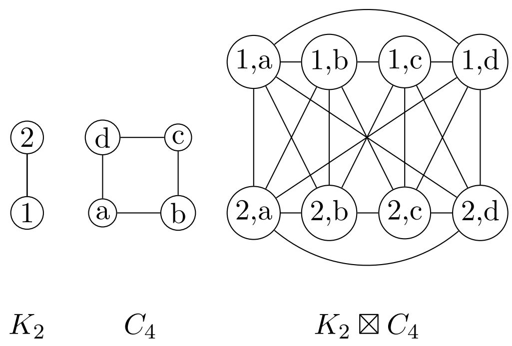

Figure 5. Strong product of K_2 and C_4

2. MAIN RESULTS

Note that in quasi-strongly regular graph with parameters (n, k, λ) we have

$$2k - \lambda = |N(u)| + |N(v)| - |N(u) \cap N(v)| = |N(u) \cup N(v)|, \qquad (*)$$

for any adjacent vertices u and v in G.

First, we prove:

Lemma 1. *Let G be a quasi-strongly regular graph with parameter (n, k, λ). If $k = \lambda + 1$ then $G = mK_{k+1}$ for some positive number m and therefore G is a strongly regular graph* $\mathrm{srg}(n, k, k-1, 0)$.

Proof. For an edge $e = (u, v)$ of G, we denote the vertices of $N(u) \cap N(v)$ by $w_1, \ldots, w_\lambda$. By $k = \lambda + 1$, $N(u) = \{v, w_1, \ldots, w_\lambda\}$ and $N(v) = \{u, w_1, \ldots, w_\lambda\}$.

Clearly, $N(w_i) = \{u, v, w_1, \ldots, w_\lambda\} \setminus \{w_i\}$. Thus, the vertices $\{v, w_1, \ldots, w_\lambda\}$ are the vertices of a complete graph with $k + 1$ vertices. Since every component of G is a complete graph K_{k+1}, we conclude that $G = mK_{k+1}$ for some positive number m. ∎

Lemma 2. *Let G be a quasi-strongly regular graph with parameter (n, k, λ). If $2k = \lambda + n$ then $G = \overline{mK_{n-k}}$ for some positive number m.*

Proof. Consider the graph $\overline{G}$. Since G is a regular graph of degree k, $\overline{G}$ is a regular graph of degree $n - 1 - k$. Let us consider two arbitrary non-adjacent vertices u and v in $\overline{G}$. In G, u and v are adjacent. By $2k = \lambda + n$ and by (*), $|N(u) \cup N(v)| = n = |V(G)|$ and therefore u and v have no common neighbor in $\overline{G}$. Thus, $\overline{G}$ is a union of disjoint complete graphs. Since $\overline{G}$ is a regular graph of degree $n - 1 - k$, $\overline{G} = mK_{n-k}$ for some positive number m. ∎

Now, we will show that the Tensor product of two quasi-strongly regular graphs is a quasi-strongly regular graph.

Theorem 1. *Let G_1 and G_2 be two quasi-strongly regular graphs with parameter (n_1, k_1, λ_1) and (n_2, k_2, λ_2), respectively. The Tensor product graph $G = G_1 \times G_2$ is a quasi-strongly regular graph with parameter $(n = n_1.n_2, k = k_1.k_2, \lambda = \lambda_1.\lambda_2)$.*

Proof. Clearly, G has $n_1.n_2$ vertices. Consider a vertex $u = (u_1, u_2)$. A vertex $v = (v_1, v_2)$ is a neighbor of u if and only if $u_1 \sim v_1$ and $u_2 \sim v_2$. Since u_1 has k_1 neighbors in G_1 and u_2 has k_2 neighbors in G_2, the number of neighbors of u in G is $k_1.k_2$.

Now we will calculate $|N(u) \cap N(v)|$ for any two adjacent vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$. By definition of Tensor product, a vertex $w = (w_1, w_2) \in N(u) \cap N(v)$ if and only if $w_1 \in N(u_1) \cap N(v_1)$ and $w_2 \in N(u_2) \cap N(v_2)$. Easy to see that the number of vertices in $N(u) \cap N(v)$ is $\lambda_1.\lambda_2$. Thus, $\lambda = \lambda_1.\lambda_2$. ■

Example 1. A cubical graph with 8 vertices (qsrg$(8, 3, 0)$) is the tensor product of K_4 (qsrg$(4, 3, 2)$) and K_2 (qsrg$(2, 1, 0)$) (see Fig. 6).

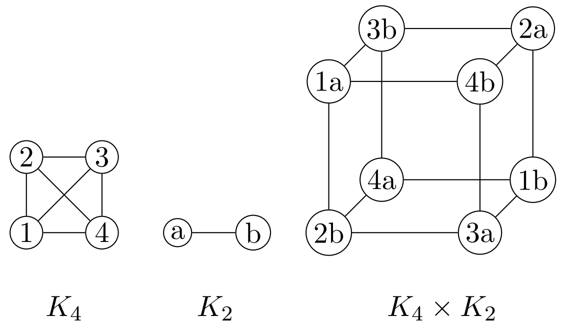

Figure 6. Tensor product of K_4 and K_2

The same result doesn't hold for Decartes product.

Theorem 2. *Let G_1 and G_2 be two quasi-strongly regular graphs with parameter (n_1, k_1, λ_1) and (n_2, k_2, λ_2), respectively. The Decartes product graph $G = G_1 \Box G_2$ is a regular graph of order $n = n_1.n_2$ and degree $k = k_1 + k_2$. Moreover, $G = G_1 \Box G_2$ is a quasi-strongly regular graph if and only if $\lambda_1 = \lambda_2$.*

Proof. Each vertex of G is an element of the Decartes product $V_1 \times V_2$, thus $n = n_1.n_2$. Consider a vertex $u = (u_1, u_2)$ in G. Its neighbors are k_2 vertices $v = (u_1, v_2)$ with $u_2 \sim v_2$ and k_1 vertices (v_1, u_2) with $u_1 \sim v_1$. Therefore, G is regular of degree $k = k_1 + k_2$.

Now we will calculate $|N(u) \cap N(v)|$ for two adjacent vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$. By the definition of the Decartes product, $u_1 = v_1 \wedge v_1 \sim v_2$ or $u_1 \sim v_1 \wedge u_2 = v_2$. We distinguish the following cases.

Case 1: $u_1 \sim v_1, u_2 = v_2$.
In this case, a vertex $w = (w_1, w_2)$ is the common neighbor of u and v only if $w_1 \in N(u_1) \cap N(v_1)$ and $w_2 = u_2 = v_2$. Thus, the number of the common neighbors of u and v is λ_1. Therefore $|N(u) \cap N(v)| = \lambda_1$.

Case 2: $u_1 = v_1$, $u_2 \sim v_2$.
Similar as in the first case, we easily get $|N(u) \cap N(v)| = \lambda_2$.

Now, we can see, the necessary and sufficient condition for the quasi-strong regularity of $G = G_1 \Box G_2$ is $\lambda_1 = \lambda_2$. ■

Example 2. The Decartes product of Paley(9) graph (qsrg$(9,4,1)$) and K_3 (qsrg$(3,2,1)$) is qsrg$(27,6,1)$ (see Fig. 7).

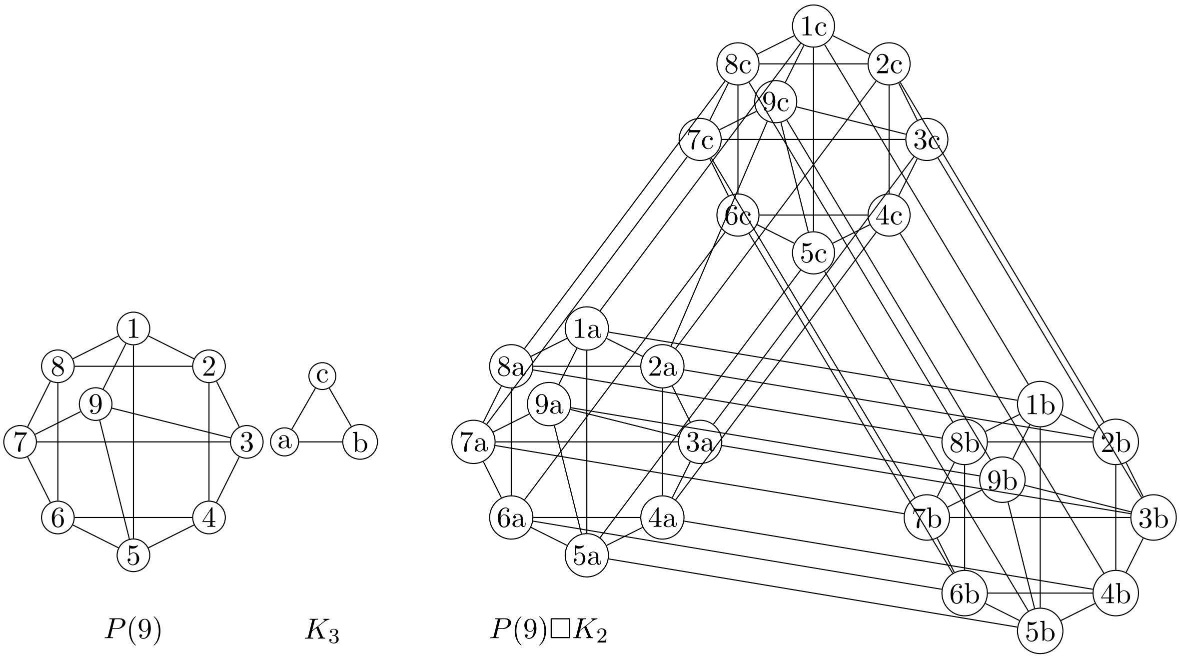

Figure 7. Decartes product of $P(9)$ and K_3

The structure of G_1, G_2 and G are more clear when we consider Lexicographical product.

Theorem 3. *Let G_1 and G_2 be two quasi-strongly regular graphs with parameter (n_1, k_1, λ_1) and (n_2, k_2, λ_2), respectively. The Lexicographical product graph $G = G_1.G_2$ is a regular graph of order $n = n_1.n_2$ and degree $k = k_1.n_2 + k_2$. The Lexicographical product graph $G = G_1.G_2$ is quasi-strongly regular graph if and only if $G_1 = pK_{k_1+1}$ and $G_2 = \overline{qK_{n_2-k_2}}$ for some positive numbers p, q.*

Proof. Every vertex $u = (u_1, u_2) \in G$ has as neighbors the vertices $v = (v_1, v_2)$ with $u_1 \sim v_1$ or $(u_1 = v_1) \wedge (u_2 \sim v_2)$. Therefore,

$$\begin{aligned} k = |N(u)| &= |\{(v_1, v_2) : v_1 \in N(u_1), v_2 \in V_2\} \cup \{(v_1, v_2) : v_1 = u_1, v_2 \in N(u_2)\}| \\ &= |\{(v_1, v_2) : v_1 \in N(u_1), v_2 \in V_2\}| + |\{(v_1, v_2) : v_1 = u_1, v_2 \in N(u_2)\}| \\ &= k_1.n_2 + k_2. \end{aligned}$$

Now, we will estimate the number of the common neighbors of two adjacent vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$. We distinguish two cases.

Case 1: $u_1 \sim v_1$.
In this case, u and v have as common neighbors the vertices $w = (w_1, w_2)$ with $w_1 \in N(u_1) \cap N(v_1)$, $w_2 \in V_2$, which has exactly $\lambda_1.n_2$ vertices, or $w_1 = u_1 \wedge w_2 \in N(u_2)$ with k_2 elements. Thus, in this case $|N(u) \cap N(v)| = \lambda_1.n_2 + 2k_2$.

Case 2: $u_1 = v_1$, $u_2 \sim v_2$.
In this case, u and v have as common neighbors the vertices $w = (w_1, w_2)$ with $w_1 = u_1 = v_1$, $w_2 \in N(u_2) \cap N(v_2)$, which has exactly λ_2 vertices, or $w_1 \in N(u_1)$, $w_2 \in V_2$, which has $k_1.n_2$ elements. Thus, in this case $|N(u) \cap N(v)| = \lambda_2 + k_1.n_2$.

The calculation for these two cases show that G is quasi-strongly regular if and only if $|N(u) \cap N(v)| = \lambda_1.n_2 + 2k_2 = \lambda_2 + k_1.n_2$ for any adjacent vertices u and v.

By $\lambda_1.n_2 + 2k_2 = \lambda_2 + k_1.n_2$, we have $2k_2 - \lambda_2 = n_2(k_1 - \lambda_1)$. By $k_i \geq \lambda_i + 1,\ i = 1, 2$ and by (*), we can easily conclude that $k_1 - \lambda_1 = 1$ and $2k_2 - \lambda_2 = n_2$. By Lemma 1, $G_1 = pK_{k_1+1}$, and by Lemma 2, $G_2 = \overline{qK_{n_2-k_2}}$. ■

Example 3. To obtain a quasi - strongly regular graph with 210 vertices, we can choose $G_1 = 2K_5$ (qsrg$(10, 4, 3)$) and $G_2 = \overline{3K_7}$ (qsrg$(21, 14, 7)$). By Theorem 3, $G = G_1.G_2$ is a qsrg$(210, 98, 91)$.

Finally, we obtain a similar result with Strong product.

Theorem 4. *Let G_1 and G_2 be two quasi-strongly regular graphs with parameter (n_1, k_1, λ_1) and (n_2, k_2, λ_2), respectively. The Strong product graph $G = G_1 \boxtimes G_2$ is a regular graph of order $n = n_1.n_2$ and degree $k = k_2 + k_1 + k_1.k_2$. The Strong product graph G is quasi-strong regular graph if and only if $G_1 = pK_{k_1+1}$ and $G_2 = qK_{k_2+1}$ for some positive numbers p, q.*

Proof. Every vertex $u = (u_1, u_2)$ in G has exactly k_1 neighbors (u_1, v_2) with $v_2 \sim u_2$, k_2 neighbors (v_1, u_2) with $v_1 \sim u_1$, $v_2 = u_2$, and k_1k_2 neighbors (v_1, v_2) where $v_1 \sim u_1$ and $v_2 \sim u_2$. Therefore $k = k_2 + k_1 + k_1.k_2$.

Consider two adjacent vertices $u = (u_1, u_2) \sim v = (v_1, v_2)$ in G. By the definition of Strong product, $u_1 = v_1 \wedge u_2 \sim v_2$ or $u_1 \sim v_1 \wedge u_2 = v_2$ or $u_1 \sim v_1 \wedge u_2 \sim v_2$. We distinguish 3 cases:

Case 1: $u_1 = v_1$, $u_2 \sim v_2$.
In this case, $N(u) \cap N(v)$ has λ_2 common neighbors (w_1, w_2) with $w_1 = u_1 = v_1$ and $w_2 \in N(u_2) \cap N(v_2)$, k_1 common neighbors (w_1, w_2) with $w_1 \in N(u_1) = N(v_1)$ and $w_2 = u_2$, and $k_1.\lambda_2$ common neighbors (w_1, w_2) with $w_1 \in N(u_1) = N(v_1)$ and $w_2 \in N(u_2) \cap N(v_2)$. Thus $\lambda = \lambda_2 + 2k_1 + k_1.\lambda_2$.

Case 2: $u_1 \sim v_1$, $u_2 = v_2$.
Similar as in Case 1, we have $\lambda = \lambda_1 + 2k_2 + k_2\lambda_1$.

Case 3: $u_1 \sim v_1, u_2 \sim v_2$.
In this case we count the number of the common neighbors similarly as in the above cases.

- Case $w_1 = u_1 \sim v_1$, $w_2 \in N(u_2) \cap N(v_2)$.

  The number of vertices of this type is λ_2.

- Case $w_1 = v_1 \sim v_1$, $w_2 \in N(u_2) \cap N(v_2)$.

  The number of vertices of this type is λ_2.

- Case $w_1 \in N(u_1) \cap N(v_1)$, $w_2 = u_2 \sim v_2$.

  The number of vertices of this type is λ_1.

- Case $w_1 \in N(u_1) \cap N(v_1)$, $w_2 = v_2 \sim v_2$.

  The number of vertices of this type is λ_1.

- Case $w_1 \in N(u_1) \cap N(v_1)$, $w_2 \in N(u_2) \cap N(v_2)$.

  The number of vertices of this type is $\lambda_1.\lambda_2$.

- Case $w_1 = u_1$, $w_2 = v_2$. The vertex $w = (u_1, v_2)$ is unique.

- Case $w_1 = v_1$, $w_2 = u_2$. The vertex $w = (v_1, u_2)$ is unique.

Therefore, in Case 3, $\lambda = 2\lambda_1 + 2\lambda_2 + \lambda_1.\lambda_2 + 2$.

Clearly, G is quasi-strongly regular if and only if

$$\lambda = \lambda_2 + 2k_1 + k_1.\lambda_2 = \lambda_1 + 2k_2 + k_2\lambda_1 = 2\lambda_1 + 2\lambda_2 + \lambda_1.\lambda_2 + 2.$$

The first equality $\lambda_2 + 2k_1 + k_1.\lambda_2 = 2\lambda_1 + 2\lambda_2 + \lambda_1.\lambda_2 + 2$ is equivalent to $(k_1 - \lambda_1 - 1)(\lambda_1 + 2) = 0$ which is equivalent to $k_1 - \lambda_1 = 1$. By Lemma 1, $G_1 = pK_{k_1+1}$. Similarly, the second equality $\lambda_1 + 2k_2 + k_2\lambda_1 = 2\lambda_1 + 2\lambda_2 + \lambda_1.\lambda_2 + 2$ implies that $G_2 = qK_{k_2+1}$. ■

We give an illustration of the above result:

Example 4. A quasi-strongly regular graph G with 770 vertices can be obtained by choosing $G_1 = 2K_{11}$ (qsrg$(22, 10, 9)$) and $G_2 = 7K_5$ (qsrg$(35, 4, 3)$). Then, $G = G_1 \boxtimes G_2$ is a quasi - strongly regular graph with $n = 770$ vertices, degree $k = 54$, $\lambda = 53$. Moreover, $G = 14K_{55}$ is a strongly regular graph with parameters $(770, 54, 53, 52)$.

3. CONCLUSION

The product of two graphs has been studied first by Vizing [12]. This method is developed in [10] by Richard Hammack, Wilfried Imrich, Sandi Klavzar. It is useful to describe graph as product of other primitive graphs. In this work, we prove some necessary and sufficient conditions for the quasi-strongly regularity of Decartes product, Tensor product, Lexicographical product and Strong product.

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